Annotation

Minimal universal crossed extensions of superperfect groups

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One of the basic facts in homological algebra is that the third cohomology group $H^3(G;M)$ classifies crossed extensions

$$0 \to M \to X \to Y \to G \to 1$$

where $\partial: X \to Y$ is a crossed module, with the resulting action of G on M again given by the G-module structure on M. This time if the action is trivial and G is superperfect, that is, both $H_1(G)$ and $H_2(G)$ are trivial, then $H^3(G;M)$ becomes isomorphic to $Hom(H_3(G),M)$.

However, the universal crossed extension of G by $H_3(G)$ is not unique anymore. There is an initial representative for it, with Y a free group and X freely (in a certain sense) built from Y and the universal cocycle on $H_3(G)$, but there are many others. In particular, it follows from the classical work of Holt that if G is finite, then there is a representative with also Y finite and X finite abelian. In fact, X can be taken the induced G-module $H_3(G)[G]$.

A natural question then arises whether there exists a smallest representative, i. e. one which is a quotient of any other representative. Indeed, we will see that for any Y-subgroup $Z \subset X$ such that $M \cap Z = 0$ one can further reduce to $X/Z \to Y/\partial(Z)$, so that if a smallest representative exists, there exists a largest Z with $M \cap Z = 0$.

Our main result is that such largest Z need not exist. Namely, we show that for G equal to the Mathieu group M_{11} there exist two distinct maximal submodules $Z \subset H_3(G)[G]$ with $H_3(G) \cap Z = 0$. As a corollary we deduce that in general there is no smallest in the above sense representative for the universal crossed extension.